

CLASSIFICATION OF THE MONOMIAL CREMONA TRANSFORMATIONS OF THE PLANE

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ABSTRACT. We classify all monomial planar Cremona maps by multidegree using recent methods developed by Aluffi. Following the main result, we prove several more properties of the set of these maps, and also extend the results to the more general ‘r.c. monomial’ maps.

1. INTRODUCTION

Let $(x : y : z)$ be coordinates on \mathbb{P}^2 . Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a monomial map

$$\varphi : (x : y : z) \mapsto (x^{a_{11}} y^{a_{12}} z^{a_{13}} : x^{a_{21}} y^{a_{22}} z^{a_{23}} : x^{a_{31}} y^{a_{32}} z^{a_{33}}).$$

We can carry the information of this map in its *exponent matrix* $M_\varphi = (a_{ij})$. We will be interested in birational monomial maps on \mathbb{P}^2 . For such maps, the total degree of each monomial is constant, i.e., there is a δ such that the row sum is δ for each row of M_φ .

If the monomials $x^{a_{i1}} y^{a_{i2}} z^{a_{i3}}$ share no common factors, we say that φ is written in *reduced form*. Throughout the paper, we’ll assume that any monomial map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ comes in reduced form. Of course, if φ is in reduced form, then M_φ must have a 0 in each column. Thus, if M_φ is the exponent matrix of a monomial Cremona transformation on \mathbb{P}^2 , then up to swapping of rows and columns it has one of the forms

$$\begin{pmatrix} 0 & 0 & \cdot \\ \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}.$$

A rational map on the projective plane has associated to it a tuple of numbers $(\gamma_0, \gamma_1, \gamma_2)$ called the multidegree (see section 3). A rational map is a Cremona map if and only if its multidegree is $(1, d, 1)$, and if the map is monomial then $d = \delta$ as above.

Theorem 1.1 below gives the complete list of monomial Cremona transformations of the plane. Notice that **(II)** could actually have been included in **(III)** if we just allowed $c = 0$. We list it separately to emphasize that it is special in the sense that the exponent matrix has more than 3 zeroes.

Theorem 1.1. *Let $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a monomial rational map with exponent matrix M_φ . Then φ is a Cremona transformation (with multidegree $(1, \delta, 1)$) if and only if*

- (I)** φ is the standard involution $\varphi(x : y : z) \mapsto (xy : xz : yz)$, or
- (II)** M_φ is of the form

$$M_\varphi = \begin{pmatrix} 0 & 0 & \delta \\ 1 & \delta - 1 & 0 \\ 0 & 1 & \delta - 1 \end{pmatrix},$$

or

(III) M_φ is of the form

$$M_\varphi = \begin{pmatrix} 0 & 0 & \delta \\ a & b & 0 \\ c & d & e \end{pmatrix}$$

and the following equations are satisfied:

- (i) $a + b = \delta$, $c + d + e = \delta$,
- (ii) $ad - bc = 1$.

Remark 1.1. Throughout the paper we will allow ourselves to swap two rows or columns of M_φ whenever convenient, as we've done here.

A useful fact about M_φ is the

Lemma 1.2. $|\det(M_\varphi)| = \delta$.

Proof. See [GSP03, Prop. 3.1], [DL, Prop. 1], [Joh, Sec. 2]. □

2. SOME CONVEX GEOMETRY

The rows of M_φ give coordinates for points in \mathbb{R}^3 .

Definition. Let $S = \{s_1, \dots, s_k\}$ be a set of points in \mathbb{R}^n . We denote the *convex hull* of S by

$$\text{ch}(S) := \{\lambda_1 s_1 + \dots + \lambda_k s_k \mid \lambda_i \geq 0, \lambda_1 + \dots + \lambda_k = 1\}$$

and the *conical hull* of S by

$$\text{cone}(S) := \{\lambda_1 s_1 + \dots + \lambda_n s_k \mid \lambda_i \geq 0\}.$$

In general, if R is an arbitrary subset of \mathbb{R}^n , we define its convex hull to be

$$\text{ch}(R) := \bigcup_{a,b \in R} \text{ch}(\{a, b\})$$

Here we are concerned with *convex polyhedra*, subsets $P \subset \mathbb{R}^n$ which can be written

$$P = \text{ch}(S) + \text{cone}(S') := \{a + b \mid a \in \text{ch}(S), b \in \text{cone}(S')\}.$$

For a polyhedron of dimension d , the faces of dimension 0, 1 and $d - 1$ are called *vertices*, *edges* and *facets*, respectively.

Let $\mathcal{B} := \{e_1, e_2, e_3\}$ be the standard basis in \mathbb{R}^3 . If

$$M_\varphi = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

is the exponent matrix of a monomial rational map φ , then the *Newton polyhedron* of φ is $N = \text{ch}(\{A, B, C\}) + \text{cone}(\mathcal{B})$. This is a 3-dimensional, unbounded polyhedron with exactly one finite facet: $\text{ch}(\{A, B, C\})$.

We will adopt the following notation for a convex polyhedron. If $S = \{v_1, \dots, v_j\}$ and $S' = \{w_1, \dots, w_k\}$ then we write $\langle v_1, \dots, v_j; w_1, \dots, w_k \rangle$ to mean $\text{ch}(S) + \text{cone}(S')$. We say that the *order* of the polyhedron is the minimal number $|S| + |S'|$. Thus, the Newton polyhedron N is $\langle A, B, C; e_1, e_2, e_3 \rangle$.

2.1. Triangulations. We will need to triangulate N . In this paper, a triangulation of a 3-dimensional convex polyhedron N is a set \mathcal{T} of 3-dimensional subsets of N such that $\sqcup_{P \in \mathcal{T}} P = N$ and such that the order of $P \in \mathcal{T}$ is 4 for all P (this makes P “simplicial”).

A useful algorithm for doing this comes from [Cla85]. We choose a distinguished vertex $p \in \{A, B, C\}$ and construct a triangulation \mathcal{T} starting with $\langle p; e_1, e_2, e_3 \rangle$, that is, the positive orthant translated to p . Now let F_p denote the facets of N which do not include the vertex p . If $f \in F_p$ is a facet with order 3, then include $\text{ch}(\{p\} \cup f)$ in \mathcal{T} . Otherwise, we should construct a triangulation \mathcal{T}_f of f using the analogous procedure and for each $t \in \mathcal{T}_f$, include $\text{ch}(\{p\} \cup t)$ in \mathcal{T} .

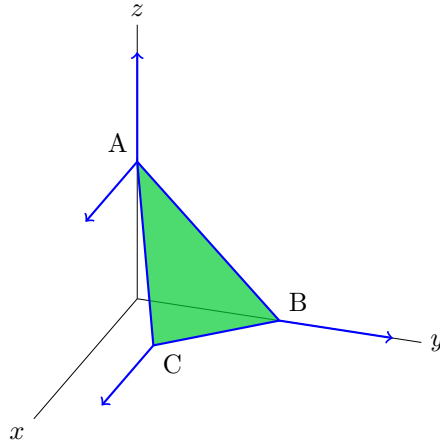


FIGURE 1. Schematic drawing of a Newton polyhedron

Example 2.1. In Figure 1, where A, B, C are given by the matrix

$$M_\varphi = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 2 & 1 & 0 \end{pmatrix},$$

we can begin a triangulation \mathcal{T} of N with $\langle A; e_1, e_2, e_3 \rangle$. We then consider the facets which do not include A . The only such face is $\langle B, C; e_1, e_2 \rangle$. Since this face has order $4 > 3$, we must triangulate it. We choose a vertex, say B , and take the cone $\langle B; e_1, e_2 \rangle$ at B . The remaining facet (of $\langle B, C; e_1, e_2 \rangle$) is $\langle C; e_2 \rangle$. Thus, the two polyhedra $\langle B; e_1, e_2 \rangle$ and $\langle B, C; e_2 \rangle$ form a triangulation of $\langle B, C; e_1, e_2 \rangle$. Taking the pyramid of these polyhedra at A completes the triangulation of N :

$$\mathcal{T} = \{\langle A; e_1, e_2, e_3 \rangle, \langle A, B; e_1, e_2 \rangle, \langle A, B, C; e_2 \rangle\}.$$

3. COMPUTING MULTIDEGREES VIA POLYHEDRA

Let $\Gamma(\varphi)$ be the closure of the graph of φ in $\mathbb{P}^2 \times \mathbb{P}^2$. Then the class $[\Gamma(\varphi)] \in A(\mathbb{P}^2 \times \mathbb{P}^2)$ is

$$[\Gamma(f)] = \gamma_0 h_2^2 + \gamma_1 h_1 h_2 + \gamma_2 h_1^2$$

where h_1, h_2 are the pullbacks of the hyperplane class $c_1(\mathcal{O}(1))$ from \mathbb{P}^2 via the projections to the first and second factors, respectively.

$$\begin{array}{ccc}
& \Gamma(\varphi) \subset \mathbb{P}^2 \times \mathbb{P}^2 & \\
\swarrow & & \searrow \\
\mathbb{P}^2 & \xrightarrow{\varphi} & \mathbb{P}^2
\end{array}$$

Definition. The *multidegree* of the rational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the tuple $(\gamma_0, \gamma_1, \gamma_2)$.

The coefficients γ_i satisfy

- (i) $\gamma_0 = \gamma_0(\varphi) := \#(\varphi^{-1}(p))$ for a general point $p \in \mathbb{P}^2$,
- (ii) $\gamma_1 = \gamma_1(\varphi)$ is the degree of (the closure of) $\varphi^{-1}(\mathbb{P}^1) \subset \mathbb{P}^2$ for a general hyperplane $H \subset \mathbb{P}^2$,
- (iii) $\gamma_2 = \gamma_2(\varphi)$ is the degree of the field extension $K(\varphi(\mathbb{P}^2)) \subset K(\mathbb{P}^2)$.

If φ is a monomial Cremona transformation, then $\gamma_0 = 1$ because φ is generically one-to-one, and $\gamma_2 = 1$ because φ is birational. The degree of the image under φ of a hyperplane in \mathbb{P}^2 will be $\#(H \cdot \varphi^{-1}(H')) = \delta$, the total degree of the monomials defining φ . The conclusion is that a monomial rational map φ on \mathbb{P}^2 is a Cremona transformation if and only if the multidegree of φ is $(1, \delta, 1)$.

3.1. Let \mathcal{T} be a triangulation of N , the Newton polyhedron of φ . If $\langle S; V \rangle \in \mathcal{T}$, we define π_V to be the projection $\mathbb{R}^3 \rightarrow V^\perp$ where V^\perp denotes the subspace spanned by $\mathcal{B} \setminus V$. We then say (in the language of [Alu, Sec. 3.1]) that the *volume* $\text{Vol}(\langle S; V \rangle)$ is the normalized volume of $\pi_V(\langle S; V \rangle)$.

Our method for computing multidegrees is [Alu, Thm. 1.4], which we restate here using our notation. Letting $T(i) = \{\langle S; V \rangle \in \mathcal{T} \mid \#(S) = i + 1\}$ be the polyhedra $\langle S; V \rangle$ whose projection under π_V are of dimension i , we have

Theorem 3.1 (Aluffi). $\gamma_i = \sum_{P \in T(i)} \text{Vol}(P)$.

Via this theorem, we have the following strategy. Find a triangulation of the Newton polyhedron N associated to an exponent matrix M_φ . If the volumes in the triangulation give $(\gamma_0, \gamma_1, \gamma_2) = (1, \delta, 1)$, then φ is a Cremona transformation, otherwise not.

4. MAIN ANALYSIS

In this section, we use the technology outlined in the previous sections to analyze the types of monomial Cremona transformations that exist, resulting in the proof of Theorem 1.1. The specific problem is: given a 3×3 integer matrix, when is it the (reduced) exponent matrix of a monomial planar Cremona transformation?

4.1. **Main case.** We begin with the case where the exponent matrix $M_\varphi = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ has the form

$$\begin{pmatrix} 0 & 0 & \cdot \\ \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & 0 & \delta \\ a & b & 0 \\ c & d & e \end{pmatrix}.$$

Note, we assume that c, d are both non-zero, since otherwise we would be in the other case (cf. section 1).

If M_φ is going to be an exponent matrix for a monomial Cremona transformation then we must have

$$(1) \quad a + b = \delta = c + d + e$$

and by Lemma 1.2 we must also have $|\delta(ad - bc)| = \delta$, so $ad - bc = \pm 1$. By swapping columns 1 and 2 if necessary, we may assume

$$(2) \quad ad - bc = 1$$

which uniquely determines the edges of the Newton polyhedron N associated to M_φ (consider the projection of the points B, C to the xy -plane), an example of which is shown in Figure 2.

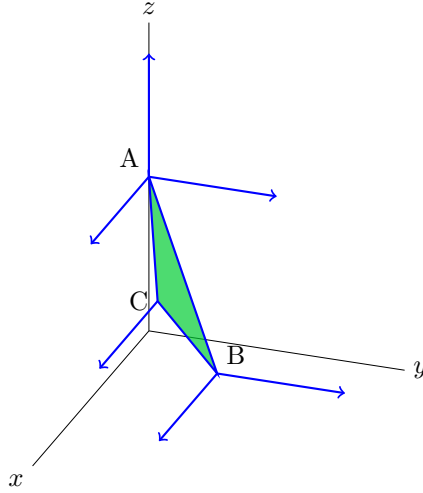


FIGURE 2

Begin constructing the triangulation \mathcal{T} with $\langle B; e_1, e_2, e_3 \rangle$. The facets of N which do not contain B are

$$\langle A; e_1, e_3 \rangle, \langle A; e_2, e_3 \rangle, \langle A, C; e_1 \rangle,$$

so we have

$$\mathcal{T} = \{ \langle B; e_1, e_2, e_3 \rangle, \langle A, B; e_1, e_3 \rangle, \langle A, B; e_2, e_3 \rangle, \langle A, B, C; e_1 \rangle \}.$$

The polyhedron $\langle B; e_1, e_2, e_3 \rangle$ projects to the point B , which has $\text{Vol}(B) = 1 = \gamma_0$. The projection of $\langle A, B; e_2, e_3 \rangle$ onto the x -axis is the interval $(0, a)$, which has $\text{Vol}((0, a)) = a$. The projection of $\langle A, B; e_1, e_3 \rangle$ onto the y -axis is the interval $(0, b)$, which has $\text{Vol}((0, b)) = b$ and as we would hope, we get $\gamma_1 = a + b = \delta$.

Finally, we consider $\langle A, B, C; e_1 \rangle$ which should be projected onto the yz -plane. The projection is a triangle with vertices $(0, \delta)$, $(b, 0)$, (d, e) , and its volume is

$$\text{Vol}(\langle A, B, C; e_1 \rangle) = |(2!)(\frac{1}{2!}) \begin{vmatrix} 0 & \delta & 1 \\ b & 0 & 1 \\ d & e & 1 \end{vmatrix}| = |\delta d + be - \delta b|$$

So since we should have $\gamma_2 = 1$ we have the necessary condition

$$(3) \quad |\delta d + be - \delta b| = 1.$$

Notice though, that if we use $a = \delta - b$ and $c = \delta - d - e$ in the determinant equation $ad - bc = 1$, we get

$$(\delta - b)d - b(\delta - d - e) = \delta d + be - \delta b = 1.$$

This shows that Equation 3 is equivalent to the requirement that $ad - bc = 1$ as long as we require also that Equation 1 be satisfied, completing the proof of the

Lemma 4.1. *If $c > 0, d > 0$, then the exponent matrix*

$$M_\varphi = \begin{pmatrix} 0 & 0 & \delta \\ a & b & 0 \\ c & d & e \end{pmatrix}$$

defines a monomial Cremona transformation with multidegree $(1, \delta, 1)$ if and only if the following equations are satisfied:

- (i) $a + b = \delta, \quad c + d + e = \delta,$
- (ii) $ad - bc = 1.$

For a quick payoff, if we set $a = \delta - 1$, then we find a uniquely determined exponent matrix:

$$M_\varphi = \begin{pmatrix} 0 & 0 & \delta \\ \delta - 1 & 1 & 0 \\ \delta - 2 & 1 & 1 \end{pmatrix}$$

and one can check that this yields a monomial Cremona transformation with inverse given by the exponent matrix

$$M_{\varphi^{-1}} = \begin{pmatrix} 1 & \delta - 1 & 0 \\ 0 & 0 & \delta \\ 1 & \delta - 2 & 1 \end{pmatrix}.$$

4.2. Other cases. Now assume M_φ has the form

$$\begin{pmatrix} 0 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}.$$

Here we have $a, b, c, d, e, f \in \mathbb{Z}_{\geq 0}$ with

$$(4) \quad a + b = c + d = e + f = \delta.$$

First note that Lemma 1.2 in this case gives $|ade + bcf| = \delta$. Rewriting using (4), we get

$$ade + bcf = a(\delta - c)e + (\delta - a)c(\delta - e) = \delta(\delta c + ae - ac - ce)$$

so

$$(5) \quad |\delta c + ae - ac - ce| = 1.$$

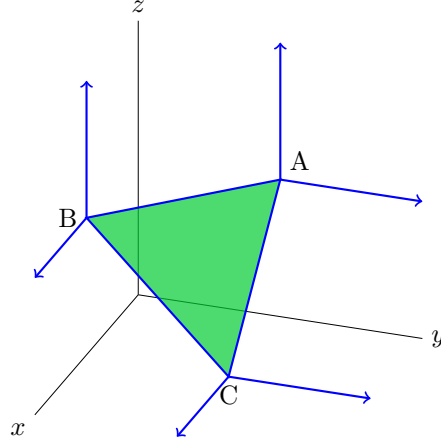


FIGURE 3. Newton polyhedron for the standard involution

4.2.1. *No additional zeroes.* Assume a, b, c, d, e, f are all nonzero. Then A is the unique minimal vertex for e_1^* , B is the unique minimal vertex for e_2 , and C the unique minimal vertex for e_3 . This determines the ridge structure of the Newton polyhedron N . An example of this structure is shown in Figure 3, where $\varphi(x : y : z) = (yz : xz : xy)$.

If we begin our triangulation with the distinguished point A , we get

$$\mathcal{T} = \{\langle A; e_1, e_2, e_3 \rangle, \langle A, B; e_1, e_3 \rangle, \langle A, C; e_2, e_3 \rangle, \langle A, B, C; e_1 \rangle\}.$$

The projection of A along all three directions e_i is a point with volume $\gamma_0 = 1$. The projection of $\text{ch}(A, B)$ along e_1, e_3 is the interval $(0, a)$, and the projection of $\text{ch}(A, C)$ along e_2, e_3 is the interval $(0, e)$. The sum of these two volumes is $\gamma_1 = a + e$. We want $\gamma_1 = \delta$, so this gives a new condition:

$$(6) \quad b = e \text{ and } a = f.$$

The projection of $\text{ch}(A, B, C)$ onto the yz -plane has normalized volume

$$\left| \det \begin{pmatrix} a & b & 1 \\ 0 & d & 1 \\ f & 0 & 1 \end{pmatrix} \right| = |ad + bf - df|$$

and we want $\gamma_2 = 1$ so we should require

$$(7) \quad |ad + bf - df| = 1$$

which becomes $|ab| = 1$ by (6). Then, $a = 1 = b$ and so $e = 1 = f$ by (6) and $c = 1 = d$ by (5). Thus,

$$M_\varphi = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

*By ‘minimal vertex for v ’, we mean $x \cdot v$ is minimized by the row vector x .

4.2.2. $f = 0$. If $f = 0$, then $e = \delta$, so we have $|ade + bcf| = |\delta ad| = \delta$ which implies $a = 1, d = 1$. Then $b = \delta - 1$ and $c = \delta - 1$, so

$$M_\varphi = \begin{pmatrix} 0 & 1 & \delta - 1 \\ \delta - 1 & 0 & 1 \\ \delta & 0 & 0 \end{pmatrix}.$$

The Newton polyhedron in this case is shown in Figure 4.

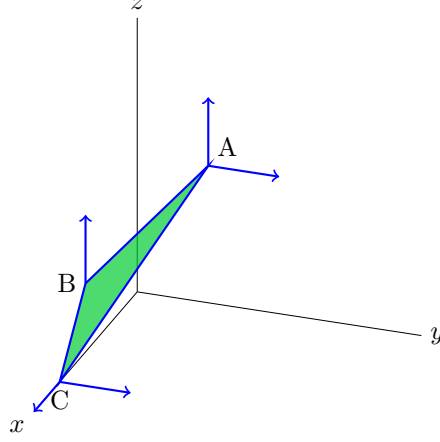


FIGURE 4. Newton polyhedron when $f = 0$

Our previous triangulation \mathcal{T} is still a triangulation here. The facet $\langle B, C; e_1, e_3 \rangle$ can be triangulated with

$$\{\langle B; e_1, e_3 \rangle, \langle B, C; e_1 \rangle\}$$

which were the facets we used before, so the algorithm still generates

$$\mathcal{T} = \{\langle A; e_1, e_2, e_3 \rangle, \langle A, B; e_1, e_3 \rangle, \langle A, C; e_2, e_3 \rangle, \langle A, B, C; e_1 \rangle\}.$$

Then to get a Cremona map, we should still require $|ad + bf - df| = 1$, and indeed this becomes

$$|1 \cdot 1 + 0 - 0| = 1.$$

This also takes care of the case where one of a, c, d is zero, since we can take the resulting matrix to one of the proper form by row and column swaps. For instance, if $d = 0$, we get

$$\begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ c & 0 & 0 \\ e & f & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & b & a \\ e & 0 & f \\ c & 0 & 0 \end{pmatrix}$$

by swapping rows 2,3 and swapping columns 2,3.

4.2.3. $e = 0$. If $e = 0$, then $f = \delta$, so we have $|ade + bcf| = |\delta bc| = \delta$, so $b = 1, c = 1$. Then $a = \delta - 1$ and $d = \delta - 1$, so

$$M_\varphi = \begin{pmatrix} 0 & \delta - 1 & 1 \\ 1 & 0 & \delta - 1 \\ 0 & \delta & 0 \end{pmatrix}.$$

But this is the same as in the previous case, just swap rows 1,2 and columns 1,2.

So this section concludes the analysis of exponent matrices which are not of the form of Lemma 4.1. The result is thus

Lemma 4.2. *Let $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a monomial rational map where M_φ is not in the form of Lemma 4.1. Then φ is a Cremona transformation (with multidegree $(1, \delta, 1)$) if and only if*

$$M_\varphi = \begin{pmatrix} 0 & \delta - 1 & 1 \\ 1 & 0 & \delta - 1 \\ 0 & \delta & 0 \end{pmatrix} \text{ or } M_\varphi = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(up to row and column swaps).

5. NUMBER OF MONOMIAL CREMONA TRANSFORMATIONS

Theorem 5.1. *For $\delta \geq 3$, there are exactly $\phi(\delta)$ exponent matrices, as described in Theorem 1.1, where ϕ is Euler's totient function.*

Proof. There is exactly one exponent matrix of the form

$$\begin{pmatrix} 0 & 0 & \delta \\ 1 & \delta - 1 & 0 \\ 0 & 1 & \delta - 1 \end{pmatrix},$$

so we must show that there are $\phi(\delta) - 1$ exponent matrices of the form described by (III) in Theorem 1.1. These are

$$\begin{pmatrix} 0 & 0 & \delta \\ a & b & 0 \\ c & d & e \end{pmatrix}$$

satisfying

- (i) $a + b = \delta, c + d + e = \delta,$
- (ii) $ad - bc = 1.$

Substituting $d = \delta - c - e$ and $b = \delta - a$ into (iii), we get

$$(8) \quad \delta a - \delta c - ae = 1$$

which we rearrange to get

$$\delta(a - c) = ae + 1.$$

This implies $\delta \mid (ae + 1)$ which we can write as $ae \equiv \delta - 1 \pmod{\delta}$. Then, for each $a \in \mathbb{Z}/\delta\mathbb{Z}$ we can set $e = a^{-1}(\delta - 1)$. However, we cannot allow $e = \delta - 1$ because this would require $c + d = 1$, which is impossible by (ii).

If a, e are determined, then $\delta r = ae + 1$ for some positive integer r , and we can let $c = r - a$.

Thus, there is a unique solution $\{a, c, e\}$ to (8) for each value of $a \in \mathbb{Z}/\delta\mathbb{Z} \setminus \{1\}$. Since this set has order $\phi(\delta) - 1$, the proof is complete. \square

By Theorem 1.1, there are exactly 2 exponent matrices having $\delta = 2$. These are

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and as always, this statement is up to row/column swapping.

We could consider these maps up to *permutation similarity*, where two exponent matrices M, M' yield the same map if and only if there exists a permutation matrix

Z such that $M' = Z^T M Z$. In other words, M, M' are equivalent if and only if there exists $\sigma \in S_3$, the symmetric group on $\{1, 2, 3\}$, such that $M = \sigma \star M'$ where S_3 acts on $M_3(\mathbb{Z})$ by permuting the rows and columns of M' . For instance, if $\sigma = (1\ 2)$, then $\sigma \star M'$ is the matrix obtained from M' by swapping the first and second rows and then swapping the first and second columns.

Since $\#(S_3) = 6$ and S_3 acts freely on the subset of $M_3(\mathbb{Z})$ defined by Theorem 1.1, under this new equivalence relation we have exactly 6 non-similar matrices for each matrix described by the theorem. So in the new context, we say that there are exactly 12 monomial Cremona transformations on \mathbb{P}^2 having $\delta = 2$.

Under this equivalence relation we have the

Corollary 5.2. *For $\delta \geq 3$ there are exactly $6 \cdot \phi(\delta)$ monomial Cremona maps on \mathbb{P}^2 up to permutation similarity.* \square

To motivate this perspective, consider the problem of determining $\gamma_1 = \delta$ for $\varphi^n = \varphi \circ \varphi \circ \cdots \circ \varphi$ for some monomial Cremona transformation φ . Let φ_1, φ_2 be the two maps defined by

$$M_1 = \begin{pmatrix} 0 & 0 & 5 \\ 4 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 5 \\ 4 & 1 & 0 \end{pmatrix},$$

that is, M_2 is given by acting with (123) on the rows of M_1 . Then, letting d_n denote $\delta = \gamma_1$ corresponding to φ^n , we have

	d_1	d_2	d_3	d_4	d_5
φ_1	5	15	40	105	275
φ_2	5	9	13	17	21

so these two maps should not be considered ‘the same’.

REFERENCES

- [Alu] Paolo Aluffi, *Multidegrees of monomial rational maps*. arXiv:1308.4152
- [Cla85] Kenneth L. Clarkson, *A probabilistic algorithm for the post office problem*, STOC (Robert Sedgewick, ed.), ACM, 1985, pp. 175–184.
- [DL] Olivier Debarre and Bodo Lass, *Monomial transformations of the projective space*. arXiv:1401.2631
- [GSP03] Gérard Gonzalez-Sprinberg and Ivan Pan, *On the monomial birational maps of the projective space*, An. Acad. Brasil. Ciênc. **75** (2003), no. 2, 129–134. MR 1984551 (2004e:14026)
- [Joh] Peter Johnson, *Inverses of monomial cremona maps*. arXiv:1105.1188

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